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Galois Module Structure of Jacobians in Unramified Extensions[1]

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For a finite unramified Galois ℓ-extension of function fields over an algebraically closed field of characteristic different from ℓ, we find the Galois module structure of the elements of the Jacobian whose orders are powers of ℓ. © 2001 Academic Press

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1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p \geq 0$. Let ℓ be a prime different from p. Let L/K be a finite Galois ℓ-extension of function fields over k with Galois group G. The group G acts naturally on the various $\mathbb{Z}_\ell$-modules associated with L, where $\mathbb{Z}_\ell$ denotes the ring of ℓ-adic integers. Let $\mathcal{J}_L$ be the corresponding Jacobian variety. We have that G acts on $\mathcal{J}_L$ and, by restriction, on ${}_{\ell^n}\mathcal{J}_L$, the group of points of $\mathcal{J}_L$ of order dividing ℓ^n. Let $\mathcal{J}_L(\ell) = \lim_{\rightarrow} {}_{\ell^n}\mathcal{J}_L$. Then $\mathcal{J}_L(\ell)$ is naturally G-isomorphic to $\mathscr{C}_{0L}(\ell)$, the ℓ-Sylow subgroup of the group $\mathscr{C}_{0L}$ of divisor classes of degree 0 of L. It is well known that as groups $\mathscr{C}_{0L}(\ell) \cong R^{2g_L}$, where g_L denotes the genus of L, $R = \mathbb{Q}_\ell/\mathbb{Z}_\ell$, and $\mathbb{Q}_\ell$ denotes the field of ℓ-adic numbers.

Our main object in this paper is to obtain explicitly the $\mathbb{Z}_\ell[G]$-module structure of $\mathcal{J}_L(\ell)$ in the case where L/K is unramified; this is Theorem 3.1.

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Our main tools are Galois cohomology, the dual of Heller's loop operator, the structure of the generalized Jacobian obtained in [8], and some results obtained in [4].

In Section 2 we collect some results from [8] and [4] and prove some results that we use later. In Section 3 we obtain the general structure of $\mathcal{J}_L(\ell)$ for L/K unramified.

When $\ell = p$, the structure of $\mathcal{J}_L(\ell)$ was found for an arbitrary extension L/K in [2]. The techniques of this paper are similar to those in [2]. However, there is a big difference: When $\ell = p$, the generalized Jacobian is $\mathbb{Z}_\ell[G]$-injective, and when $\ell \neq p$, the generalized Jacobian is not $\mathbb{Z}_\ell[G]$-injective ([8]). The reason is that when $\ell \neq p$, the ℓ^nth roots of unity are present in the base field. The case where $\ell \neq p$, L/K ramified, remains to be solved.

2. RESULTS ON COHOMOLOGY AND NOTATIONS

In this paper, $\mathcal{D}_L$ denotes the group of divisors of the field L; $\mathcal{D}_{0L}$, the subgroup of $\mathcal{D}_L$ of divisors of degree 0; P_L, the principal divisors; $\mathcal{C}_L$, the class group of L; and $\mathcal{C}_{0L}$, the group of divisor classes of degree 0. For a natural number m, C_m denotes a cyclic group of order m. We denote by G the Galois group $\mathrm{Gal}(L/K)$ of the ℓ-extension L/K, ℓ different from the characteristic. Let $[G] = \ell^n$ be the order of G, and let g_L denote the genus of L.

For any G-module M, the ith Tate cohomology group $\widehat{H}^i(G, M)$, $i \in \mathbb{Z}$, is denoted by $H^i(G, M)$. The trivial group is denoted by 0, whether its structure is additive or multiplicative. Also, we denote by $M^G = \{m \in M \mid gm = m \;\forall\, g \in G\}$ and by $N_G = N$ the norm (trace) map, that is, $Nm = \prod_{g\in G} gm$ (or $\sum_{g\in G} gm$), ${}_N M$ denotes the kernel of N acting on M and $I_G M$ denotes the module generated by $\langle gm - m \mid g \in G, m \in M\rangle$.

For any $\mathbb{Z}_\ell[G]$-module M, $\mathcal{X}(M)$ denotes the Pontrjagin dual of M, that is, $\mathcal{X}(M) = \mathrm{Hom}_{\mathbb{Z}_\ell}(M, R)$. Finally, for any module M, ${}_\ell M$ denotes the elements of M whose order divide ℓ.

Each module under consideration is finitely generated, or its Pontrjagin dual is finitely generated.

First, we consider the following exact sequences of $\mathbb{Z}_\ell[G]$-modules:

$$0 \longrightarrow k^* \longrightarrow L^* \longrightarrow P_L \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow P_L \longrightarrow \mathcal{D}_L \longrightarrow \mathcal{C}_L \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow P_L \xrightarrow{i} \mathcal{D}_{0L} \xrightarrow{\pi} \mathcal{C}_{0L} \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow \mathcal{D}_{0L} \longrightarrow \mathcal{D}_L \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow \mathscr{C}_{0L} \longrightarrow \mathscr{C}_L \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0, \tag{5}$$

and

$$0 \longrightarrow \mathbb{Z}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow R \longrightarrow 0, \tag{6}$$

where deg denotes the degree map.

Let L/K be an unramified extension. Then we have $H^0(G, \mathscr{D}_L) \cong H^{-1}(G, \mathscr{D}_L) = \{0\}$ and, by Tsen's theorem, $H^0(G, L^*) = \{0\}$. From Hilbert's theorem 90, we obtain that $H^1(G, L^*) = \{0\}$. Since G is an ℓ-group and the cohomology groups vanish for two consecutive values, it follows that L^* and $\mathscr{D}_L$ are cohomologically trivial ([6], Theorem 8, Chapter IX, Section 5). We also have that $\mathbb{Q}_\ell$ is cohomologically trivial.

Now, since G is an ℓ-group and k is an algebraically closed field, we have that $k^*(\ell) = \{\xi \in k \mid \xi^{\ell^n} = 1 \text{ for some } n \in \mathbb{N}\} \cong R$ and

$$H^i(G, k^*) \cong H^i(G, k^*(\ell)) \cong H^i(G, R). \tag{7}$$

From (1), (6), and (7), it follows that

$$\begin{aligned} H^i(G, P_L) &\cong H^{i+1}(G, k^*) \cong H^{i+1}(G, R) \\ &\cong H^{i+2}(G, \mathbb{Z}_\ell) \cong H^{i+2}(G, \mathbb{Z}), \qquad i \in \mathbb{Z}. \end{aligned} \tag{8}$$

From (2) and (8), we obtain that

$$H^i(G, \mathscr{C}_L) \cong H^{i+1}(G, P_L) \cong H^{i+3}(G, \mathbb{Z}), \qquad i \in \mathbb{Z}. \tag{9}$$

From (4), it follows that

$$H^i(G, \mathbb{Z}) \cong H^{i+1}(G, \mathscr{D}_{0L}). \tag{10}$$

We have that ([10], Corollary 4.4.7)

$$H^i(G, \mathbb{Z}) \cong H^{-1}(G, \mathbb{Z}) \text{ for all } i \in \mathbb{Z}, \tag{11}$$

$$H^1(G, \mathbb{Z}) \cong H^{-1}(G, \mathbb{Z}) = \{0\}, \tag{12}$$

$$H^0(G, \mathbb{Z}) \cong C_{|G|}, \tag{13}$$

and

$$H^{-2}(G, \mathbb{Z}) \cong H^2(G, \mathbb{Z}) \cong G/G', \tag{14}$$

where G' denotes the commutator subgroup of G.

From (3), we obtain

$$\begin{aligned} \cdots \longrightarrow H^{-1}(G, P_L) \longrightarrow H^{-1}(G, \mathscr{D}_{0L}) &\xrightarrow{f} H^{-1}(G, \mathscr{C}_{0L}) \\ \xrightarrow{g} H^0(G, P_L) \longrightarrow H^0(G, \mathscr{D}_{0L}) &\longrightarrow \cdots. \end{aligned} \tag{15}$$

From (8), (12), and (14), we have that

$$H^{-1}(G, P_L) = \{0\} \qquad \text{and} \qquad H^0(G, P_L) \cong G/G'.$$

From (10), (12), and (14), it follows that

$$H^{-1}(G, \mathcal{D}_{0L}) \cong G/G' \qquad \text{and} \qquad H^0(G, \mathcal{D}_{0L}) = \{0\}.$$

Therefore,

$$0 \longrightarrow G/G' \longrightarrow H^{-1}(G, \mathcal{C}_{0L}) \longrightarrow G/G' \longrightarrow 0 \tag{16}$$

is exact.

We will see that (16) splits. From Tsen's theorem, we have that $NL = K$. Therefore, $NP_L = P_K$. From (15), we obtain

$$0 \longrightarrow \frac{{}_N\mathcal{D}_{0L}}{I_G\mathcal{D}_{0L}} \xrightarrow{f} \frac{{}_N\mathcal{C}_{0L}}{I_G\mathcal{C}_{0L}} \xrightarrow{g} \frac{P_L^G}{P_K} \longrightarrow 0. \tag{17}$$

Now f is given by $f(\mathcal{A} \bmod I_G\mathcal{D}_{0L}) = \pi(\mathcal{A}) \bmod I_G\mathcal{C}_{0L}$, π and i are the maps given in (3), and g is the connecting homomorphism given as follows: If $c \in {}_N\mathcal{C}_{0L}(\ell)$, then let $\mathfrak{A} \in \mathcal{D}_{0L}$ such that $\pi(\mathfrak{A}) = c$. Then $0 = Nc = N\pi\mathfrak{A} = \pi(N\mathfrak{A})$. It follows that $N\mathfrak{A} \in \ker \pi = \operatorname{im} i$; that is, there exists $a \in L^*, (a) \in P_L$, such that $i((a)) = N\mathfrak{A}$. Then $g(c \bmod I_G\mathcal{C}_{0L}) = (a) \bmod P_K$ (symbolically, $g(\bar{c}) = i^{-1}N\pi^{-1}(c) \bmod P_K$).

We consider the map $\delta : \frac{P_L^G}{P_K} \to \frac{{}_N\mathcal{C}_{0L}}{I_G\mathcal{C}_{0L}}$ given as follows: If $(a) \in P_L^G \subseteq \mathcal{D}_{0L}^G = N\mathcal{D}_{0L} = \mathcal{D}_{0K}$ (since $H^0(G, \mathcal{D}_{0L}) = \{0\}$), then there exists $\mathfrak{A} \in \mathcal{D}_{0L}$ such that $N\mathfrak{A} = (a)$. Let

$$\delta((a) \bmod P_K) = \pi(\mathfrak{A}) \bmod I_G\mathcal{C}_{0L}. \tag{18}$$

If δ is well defined, then it follows immediately that $g \circ \delta = \mathrm{Id}_{H^0(G,P_L)}$ and that (16) splits.

To show that δ is well defined we must prove that if $\mathfrak{A}, \mathfrak{B} \in \mathcal{D}_{0L}$ are such that $N\mathfrak{A} = N\mathfrak{B} = (a)$, then $\pi(\mathfrak{A}) \equiv \pi(\mathfrak{B}) \bmod I_G\mathcal{C}_{0L}$.

We have the exact sequence

$$0 \longrightarrow {}_N\mathcal{D}_{0L} \longrightarrow \mathcal{D}_{0L} \xrightarrow{\theta=N} N\mathcal{D}_{0L} = \mathcal{D}_{0K} \longrightarrow 0.$$

Therefore,

$$0 \longrightarrow \frac{{}_N\mathcal{D}_{0L}}{I_G\mathcal{D}_{0L}} \longrightarrow \frac{\mathcal{D}_{0L}}{I_G\mathcal{D}_{0L}} \xrightarrow{\bar{\theta}} N\mathcal{D}_{0L} = \mathcal{D}_{0K} \longrightarrow 0 \tag{19}$$

is an exact sequence of groups.

Since $\mathcal{D}_{0K}$ is a $\mathbb{Z}$-free module, (19) splits. Furthermore, the splitting map ε: $N\mathcal{D}_{0L} \to \frac{\mathcal{D}_{0L}}{I_G\mathcal{D}_{0L}}$ is given as follows: If $\mathfrak{D} \in \mathcal{D}_{0L}$, then $\varepsilon(N\mathfrak{D}) = \mathfrak{D} \bmod I_G\mathcal{D}_{0L}$.

Therefore, if $N\mathfrak{A} = N\mathfrak{B}$, then it follows that $\mathfrak{A} \equiv \mathfrak{B} \bmod I_G\mathcal{D}_{0L}$; that is, there exists $\mathfrak{C} \in I_G\mathcal{D}_{0L}$ such that $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$. Hence $\pi(\mathfrak{A}) = \pi(\mathfrak{B})\pi(\mathfrak{C})$ and $\pi(\mathfrak{C}) \in I_G\mathcal{C}_{0L}$.

Thus, δ is well defined and

$$H^{-1}(G, \mathcal{C}_{0L}) \cong G/G' \oplus G/G'. \tag{20}$$

Now we study the group $H^0(G, \mathcal{C}_{0L})$. We want to prove that

$$H^0(G, \mathcal{C}_{0L}) \cong H^0(G, \mathcal{C}_L).$$

From [4], we have that $H^0(H, \mathcal{C}_{0L}) \cong H^0(H, \mathcal{C}_L) \cong \{0\}$ for any $H < G$ cyclic. Let H be a cyclic subgroup contained in the center of G of order ℓ. Let $E = L^H$ be the fixed field, $\mathrm{Gal}(L/E) = H$.

From (2), (3), and [4], we obtain

$$0 \longrightarrow P_L^H \longrightarrow \mathcal{D}_L^H \longrightarrow \mathcal{C}_L^H \longrightarrow H^1(H, P_L) = \{0\} \tag{21}$$

and

$$0 \longrightarrow P_L^H \longrightarrow \mathcal{D}_{0L}^H \longrightarrow \mathcal{C}_{0L}^H \longrightarrow H^1(H, P_L) = \{0\}. \tag{22}$$

Since $\mathcal{D}_L^H = \mathcal{D}_E$ and $\mathcal{D}_{0L}^H = \mathcal{D}_{0E}$, it follows from (21) and (22) that

$$0 \longrightarrow P_L^H/P_E \longrightarrow \mathcal{C}_E \longrightarrow \mathcal{C}_L^H \longrightarrow 0 \tag{23}$$

and

$$0 \longrightarrow P_L^H/P_E \longrightarrow \mathcal{C}_{0E} \longrightarrow \mathcal{C}_{0L}^H \longrightarrow 0. \tag{24}$$

Now $A := P_L^H/P_E \cong H^0(H, P_L) \cong C_\ell$ (see [4]) and

$$H^0(G/H, A) \cong H^1(G/H, A) \cong A. \tag{25}$$

Now from [9], Theorem 1, and [4] we obtain the exact sequences

$$H^0(H, \mathcal{C}_L) \cong 0 \longrightarrow H^0(G, \mathcal{C}_L) \longrightarrow H^0(G/H, \mathcal{C}_L^H) \longrightarrow 0 \tag{26}$$

and

$$H^0(H, \mathcal{C}_{0L}) \cong 0 \longrightarrow H^0(G, \mathcal{C}_{0L}) \longrightarrow H^0(G/H, \mathcal{C}_{0L}^H) \longrightarrow 0. \tag{27}$$

Therefore,

$$H^0(G, \mathcal{C}_L) \cong H^0(G/H, \mathcal{C}_L^H) \quad \text{and} \quad H^0(G, \mathcal{C}_{0L}) \cong H^0(G/H, \mathcal{C}_{0L}^H). \tag{28}$$

By induction on the order of G, we assume that

$$H^0(G/H, \mathcal{C}_E) \cong H^0(G/H, \mathcal{C}_{0E}).$$

From (23), (24), and (25), we obtain the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\jmath} & H^0(G/H, \mathscr{C}_E) & \xrightarrow{\pi} & H^0(G/H, \mathscr{C}_L^H) & \xrightarrow{\delta} & A \\ \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \mathrm{Id}} \\ A & \xrightarrow{\jmath'} & H^0(G/H, \mathscr{C}_{0E}) & \xrightarrow{\pi'} & H^0(G/H, \mathscr{C}_{0L}^H) & \xrightarrow{\delta'} & A \end{array} \tag{29}$$

where $\jmath, \jmath', \pi, \pi', \alpha$, and β are the natural maps and δ and δ' are the connecting homomorphisms.

Since α is an isomorphism, it follows that β is an isomorphism. Therefore, from (28), we have

$$H^0(G, \mathscr{C}_{0L}) \cong H^0(G, \mathscr{C}_L). \tag{30}$$

Therefore, from (5), (12), and (30), we obtain

$$\begin{aligned} H^{-1}(G, \mathbb{Z}) = \{0\} &\longrightarrow H^0(G, \mathscr{C}_{0L}) \longrightarrow H^0(G, \mathscr{C}_L) \xrightarrow{\widetilde{\deg}} H^0(G, \mathbb{Z}) \\ &\cong \mathbb{Z}/\ell^n\mathbb{Z} \longrightarrow H^1(G, \mathscr{C}_{0L}) \longrightarrow H^1(G, \mathscr{C}_L) \longrightarrow \{0\} \\ &= H^1(G, \mathbb{Z}). \end{aligned} \tag{31}$$

Thus, $\widetilde{\deg}\colon H^0(G, \mathscr{C}_L) = \mathscr{C}_L^G/N\mathscr{C}_L \to H^0(G, \mathbb{Z}) \cong \mathbb{Z}/\ell^n\mathbb{Z}$ is the zero map.

It follows from (31) that

$$0 \longrightarrow H^0(G, \mathscr{C}_{0L}) \longrightarrow H^0(G, \mathscr{C}_L) \longrightarrow 0 \tag{32}$$

and

$$0 \longrightarrow C_{\ell^n} \longrightarrow H^1(G, \mathscr{C}_{0L}) \longrightarrow H^1(G, \mathscr{C}_L) \longrightarrow 0 \tag{33}$$

are exact sequences of groups.

Finally, since $\ell^n H^1(G, \mathscr{C}_{0L}) = 0$, from (9), (20), (30), and (33), we obtain

PROPOSITION 2.1. *Let L/K be an unramified Galois ℓ-extension of degree ℓ^n. Then the cohomology groups of $\mathscr{C}_{0L}(\ell)$ are given by*:

(i) $H^{-1}(G, \mathscr{C}_{0L}(\ell)) \cong G/G' \oplus G/G'$,
(ii) $H^0(G, \mathscr{C}_{0L}(\ell)) \cong H^3(G, \mathbb{Z})$, *and*
(iii) $H^1(G, \mathscr{C}_{0L}(\ell)) \cong C_{\ell^n} \oplus H^4(G, \mathbb{Z})$.

Let M be a $\mathbb{Z}_\ell[G]$-module that is $\mathbb{Z}_\ell$-divisible and such that $\mathscr{X}(M)$ is finitely generated. That is, as groups, $M \cong R^s$ for some $s \in \mathbb{N} \cup \{0\}$. Then

$$0 \longrightarrow {}_\ell M \longrightarrow M \xrightarrow{\ell} M \longrightarrow 0 \tag{34}$$

is an exact $\mathbb{Z}_\ell[G]$-sequence.

From (34), we obtain the exact sequence

$$\cdots \longrightarrow H^{i-1}(G, M) \xrightarrow{\ell} H^{i-1}(G, M) \longrightarrow H^i(G, {}_\ell M) \longrightarrow H^i(G, M) \xrightarrow{\ell} H^i(G, M) \longrightarrow \cdots. \tag{35}$$

It follows that

$$H^i(G, {}_\ell M) \cong C_\ell^{\alpha_{i-1}(M)+\alpha_i(M)}, \tag{36}$$

where

$$\alpha_i(M) = \dim_{\mathbb{F}_\ell} \frac{H^i(G, M)}{\ell H^i(G, M)} = \dim_{\mathbb{F}_\ell} {}_\ell H^i(G, M),$$

with $\mathbb{F}_\ell$ denoting the finite field with ℓ elements.

Now we have that (see [5])

$$\alpha_2(\mathbb{Z}) = d \qquad \text{and} \qquad \alpha_3(\mathbb{Z}) = r - d, \tag{37}$$

where d is the minimal number of generators of G and r is the number of relations.

From Proposition 2.1 and (37), we obtain that $\alpha_{-1}(\mathscr{C}_{0L}) = 2d$ and $\alpha_0(\mathscr{C}_{0L}) = \alpha_3(\mathbb{Z}) = r - d$. Hence, from (36), we have

PROPOSITION 2.2. *With the hypothesis of Proposition* 2.1, *we have that*

$$H^0(G, {}_\ell\mathscr{C}_{0L}) \cong C_\ell^{d+r}.$$

3. GALOIS MODULE STRUCTURE OF $\mathscr{J}_L(\ell)$

We keep the notation and hypothesis of Section 2. Let $\wp$ be a fixed prime divisor of K and let $\mathscr{N} = \operatorname{con}_{K/L}\wp = \prod_{i=1}^{|G|} \mathfrak{P}_i$ be the conorm of $\wp$ in L. We have the $\mathbb{Z}_\ell[G]$-exact sequence (see [8])

$$0 \longrightarrow T \longrightarrow \mathscr{C}_{0\mathscr{N}}(\ell) \longrightarrow \mathscr{C}_{0L}(\ell) \longrightarrow 0, \tag{38}$$

where $\mathscr{C}_{0\mathscr{N}}(\ell) \cong \mathscr{J}_{\mathscr{N}}(\ell)$, $\mathscr{J}_{\mathscr{N}}(\ell)$ is the ℓ-torsion of the generalized Jacobian associated to L and $\mathscr{N}$, and $T \cong \frac{R[G]}{R}$.

We have that

$$\mathscr{C}_{0\mathscr{N}}(\ell) \cong R[G]^{2g_{K-d}} \oplus S \tag{39}$$

where $S \cong R^{|G|(d-1)+1}$ as groups and S is an indecomposable $\mathbb{Z}_\ell[G]$-module ([8], Theorem 6).

We also have ([8])

$$H^i(G, \mathscr{C}_{0\mathcal{N}}(\ell)) \cong H^i(G, S) \cong H^{i-1}(G, \mathbb{Z}). \tag{40}$$

From the exact sequence

$$0 \longrightarrow R \longrightarrow R[G] \longrightarrow T \longrightarrow 0, \tag{41}$$

we obtain that

$$0 \to R \to R[G]^G \cong R \to T^G \to H^1(G, R) \to 0$$

is exact and that

$$H^i(G, T) \cong H^{i+1}(G, R) \cong H^{i+2}(G, \mathbb{Z}). \tag{42}$$

Therefore,

$$T^G = H^0(G, T) \cong G/G' \qquad \text{and} \qquad NT = 0.$$

Now, taking the elements of order dividing ℓ in (38), we obtain the $\mathbb{F}_\ell[G]$-exact sequence

$$0 \longrightarrow {}_\ell T \cong \frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell} \xrightarrow{\delta} {}_\ell\mathscr{C}_{0\mathcal{N}} \longrightarrow {}_\ell\mathscr{C}_{0L} \longrightarrow 0. \tag{43}$$

From (43), we obtain the exact sequence

$$\begin{aligned} H^{-1}(G, {}_\ell\mathscr{C}_{0L}) \longrightarrow H^0(G, {}_\ell T) \longrightarrow H^0(G, {}_\ell\mathscr{C}_{0\mathcal{N}}) \xrightarrow{\psi} H^0(G, {}_\ell\mathscr{C}_{0L}) \\ \longrightarrow H^1(G, {}_\ell T) \longrightarrow H^1(G, {}_\ell\mathscr{C}_{0\mathcal{N}}). \end{aligned} \tag{44}$$

Since ${}_\ell T \cong \frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell}$ and $N(\mathbb{F}_\ell[G]) = \mathbb{F}_\ell$, it follows that $N({}_\ell T) = 0$.
From (11), (12), (13), (37), and (40), we have that

$$\alpha_{-1}(\mathscr{C}_{0\mathcal{N}}) = d, \alpha_0(\mathscr{C}_{0\mathcal{N}}) = 0 \qquad \text{and} \qquad \alpha_1(\mathscr{C}_{0\mathcal{N}}) = 1. \tag{45}$$

From (45) and (36), we have that

$$H^0(G, {}_\ell\mathscr{C}_{0\mathcal{N}}) \cong C_\ell^{d+0} \cong C_\ell^d \tag{46}$$

and

$$H^1(G, {}_\ell\mathscr{C}_{0\mathcal{N}}) \cong C_\ell^{0+1} \cong C_\ell. \tag{47}$$

From (42), we obtain that

$$\alpha_0(T) = d \qquad \text{and} \qquad \alpha_1(T) = r - d. \tag{48}$$

From (36) and (48), it follows that

$$H^1(G, {}_\ell T) \cong C_\ell^{d+(r-d)} \cong C_\ell^r. \tag{49}$$

From (44), (46), (47), and (49), we obtain that

$$H^{-1}(G, {}_\ell\mathscr{C}_{0L}) \longrightarrow {}_\ell T^G \xrightarrow{\delta} H^0(G, {}_\ell\mathscr{C}_{0\mathcal{N}}) \cong C_\ell^d \xrightarrow{\psi} H^0(G, {}_\ell\mathscr{C}_{0L})$$
$$\cong C_\ell^{d+r} \longrightarrow H^1(G, {}_\ell T) \cong C_\ell^r \xrightarrow{\phi} H^1(G, {}_\ell\mathscr{C}_{0\mathcal{N}}) \cong C_\ell$$

is exact.

Therefore,

$$0 \longrightarrow \frac{H^0(G, {}_\ell\mathscr{C}_{0\mathcal{N}})}{\ker\psi} \longrightarrow H^0(G, {}_\ell\mathscr{C}_{0L}) \longrightarrow \ker\phi \longrightarrow 0$$

is exact.

Since $H^0(G, {}_\ell\mathscr{C}_{0\mathcal{N}}) \cong C_\ell^d$, $H^0(G, {}_\ell\mathscr{C}_{0L}) \cong C_\ell^{d+r}$ and $\ker\phi \subseteq H^1(G, {}_\ell T) \cong C_\ell^r$, it follows that $\ker\psi = \operatorname{im}\delta = \{0\}$ and $\ker\phi = H^1(G, {}_\ell T)$.

In particular,

$$ {}_\ell T^G \subseteq N({}_\ell\mathscr{C}_{0\mathcal{N}}). \tag{50}$$

Since S has no $\mathbb{Z}_\ell[G]$-injective components, we have that $N({}_\ell S) = 0$.

Assume that $T \cap S \neq \{0\}$. Then

$$ {}_\ell(T \cap S) = {}_\ell T \cap {}_\ell S \neq \{0\}. \tag{51}$$

Since ${}_\ell(T \cap S)$ is a finite nontrivial ℓ-group, we have ${}_\ell T^G \cap {}_\ell S^G = ({}_\ell T \cap {}_\ell S)^G \neq \{0\}$.

Now we have that $N({}_\ell\mathscr{C}_{0\mathcal{N}}) = N({}_\ell M)$, where $\mathscr{C}_{0\mathcal{N}} \cong M \oplus S$ and $M \cong \mathscr{C}_{0\mathcal{N}}/S \cong R[G]^{2g_K - d}$.

It follows that

$$ {}_\ell T^G \subseteq N({}_\ell\mathscr{C}_{0\mathcal{N}}) = N({}_\ell M) \subseteq {}_\ell M. \tag{52}$$

From (51) and (52), we obtain that

$$\{0\} = {}_\ell\{0\} = {}_\ell(M \cap S) = {}_\ell M \cap {}_\ell S \supseteq {}_\ell T^G \cap {}_\ell S^G \neq \{0\}.$$

This contradiction shows that

$$T \cap S = \{0\}. \tag{53}$$

Therefore, we have

THEOREM 3.1. *Let L/K be a finite unramified Galois ℓ-extension of function fields with Galois group G. Then the $\mathbb{Z}_\ell[G]$-module structure of $\mathscr{C}_{0L}(\ell)$, in terms of indecomposable modules is given by*

$$\mathscr{C}_{0L}(\ell) \cong R[G]^{2g_K - 2d} \oplus \Omega^{\#}(T) \oplus S, \tag{54}$$

where $\Omega^{\#}(T) \cong R[G]^d/(R[G]/R)$.

Furthermore, $\Omega^{\#}(T)$ and S are indecomposable $\mathbb{Z}_\ell[G]$-modules such that, as groups, $\Omega^{\#}(T) \cong S \cong R^{|G|(d-1)+1}$.

Proof. From (38) and (53), we have that the natural map $\rho : T \to \frac{\mathscr{C}_{0\mathcal{N}}(\ell)}{S}$ is an injective homomorphism. From (39), we have that $\frac{\mathscr{C}_{0\mathcal{N}}(\ell)}{S} \cong R[G]^{2g_K-d}$. Therefore, there exists a decomposition $\mathscr{C}_{0\mathcal{N}}(\ell) \cong M \oplus S$, $M \cong R[G]^{2g_K-d}$, such that in the $\mathbb{Z}_\ell[G]$-exact sequence

$$0 \longrightarrow T \xrightarrow{j} M \oplus S \longrightarrow \mathscr{C}_{0\mathcal{N}}(\ell) \longrightarrow 0,$$

$j(T) \subseteq M$ and S is indecomposable ([8]). Therefore,

$$\mathscr{C}_{0L}(\ell) \cong \frac{R[G]^{2g_K-d}}{j(T)} \oplus S.$$

Let $\Omega^{\#}$ denote the dual of the Heller's loop operator (see [2]). Then d is the minimal natural number such that there exists a $\mathbb{Z}_\ell[G]$-injective homomorphism $\chi: T \to R[G]^d$. Therefore,

$$\Omega^{\#}(T) \cong \frac{R[G]^d}{T} \cong \frac{R[G]^d}{(R[G]/R)},$$

and

$$0 \longrightarrow T \longrightarrow R[G]^{2g_K-d} \longrightarrow \Omega^{\#}(T) \oplus R[G]^{2g_K-2d} \longrightarrow 0$$

is exact. Therefore,

$$\mathscr{C}_{0L}(\ell) \cong \frac{R[G]^{2g_K-d}}{j(T)} \oplus S \cong R[G]^{2g_K-2d} \oplus \Omega^{\#}(T) \oplus S.$$

Finally, since T is indecomposable, we have that $\Omega^{\#}(T)$ is an indecomposable module and, as groups, $\Omega^{\#}(T) \cong R^{|G|(d-1)+1} \cong S$. ∎

Remark 3.1. We have that $\Omega^{\#}(T) \cong S$ as $\mathbb{Z}_\ell[G]$-modules iff $d = 1$, that is, only in the case where G is a cyclic group. In this case we have $\Omega^{\#}(T) \cong S \cong R$. In fact, we have (see (40))

$$H^i(G, S) \cong H^i(G, \mathscr{C}_{0\mathcal{N}}(\ell)) \cong H^{i-1}(G, \mathbb{Z}),$$

and from the exact sequence

$$0 \longrightarrow T \longrightarrow R[G]^d \longrightarrow \Omega^{\#}(T) \longrightarrow 0 \tag{55}$$

and (42), we have that $H^i(G, \Omega^{\#}(T)) \cong H^{i+1}(G, T) \cong H^{i+3}(G, \mathbb{Z})$.

In particular, if $S \cong \Omega^{\#}(T)$, then

$$H^i(G, S) \cong H^{i-1}(G, \mathbb{Z}) \cong H^{i+3}(G, \mathbb{Z}) \cong H^i(G, \Omega^{\#}(T)).$$

In particular, $H^4(G, \mathbb{Z}) \cong H^0(G, \mathbb{Z}) \cong C_{|G|}$. This occurs only in the case where G is a cyclic group (see [1], Theorem 9.5, Chapter VI, and Exercise 4, p. 159). In this situation, $d = 1$ and $S \cong R \cong \Omega^{\#}(T)$. This result was obtained in [4].

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